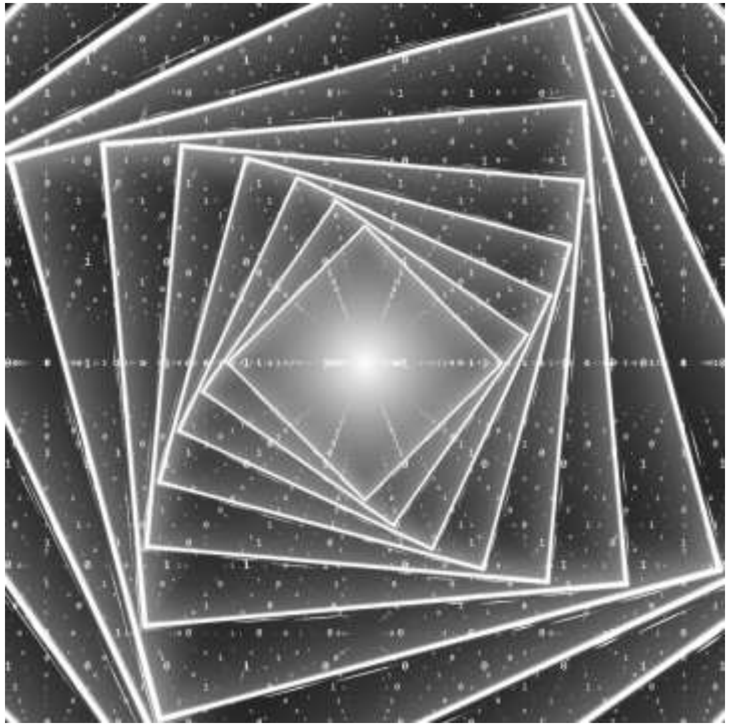
Sequences

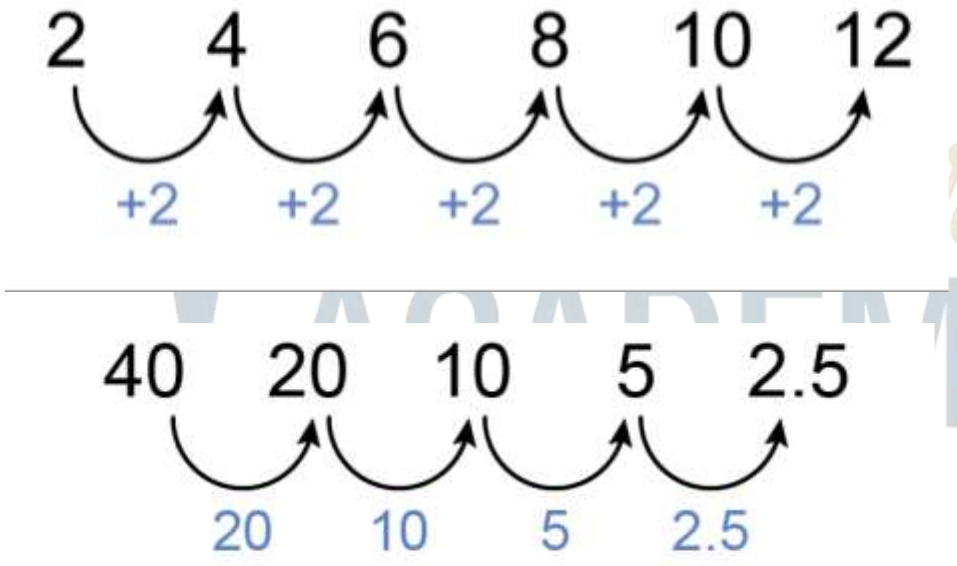


1 Historical view

Mathematicians throughout history have been fascinated by patterns in nature and numbers. The famous Fibonacci sequence, for example, appears in everything from seashells to galaxies, revealing a fundamental order in the universe.

2 Definition

A **sequence** is an ordered list of numbers.
Unlike a set, the order of the numbers in a sequence matters, and the same number can appear multiple times.
Many sequences follow specific rule or formula that can be used to find any term in the sequence.



2 Definition

Each number is called "term".

We often use subscript notation to represent the terms.

Example:

- First term: a_1 or u_1 …
- Second term: a_2 or u_2 …

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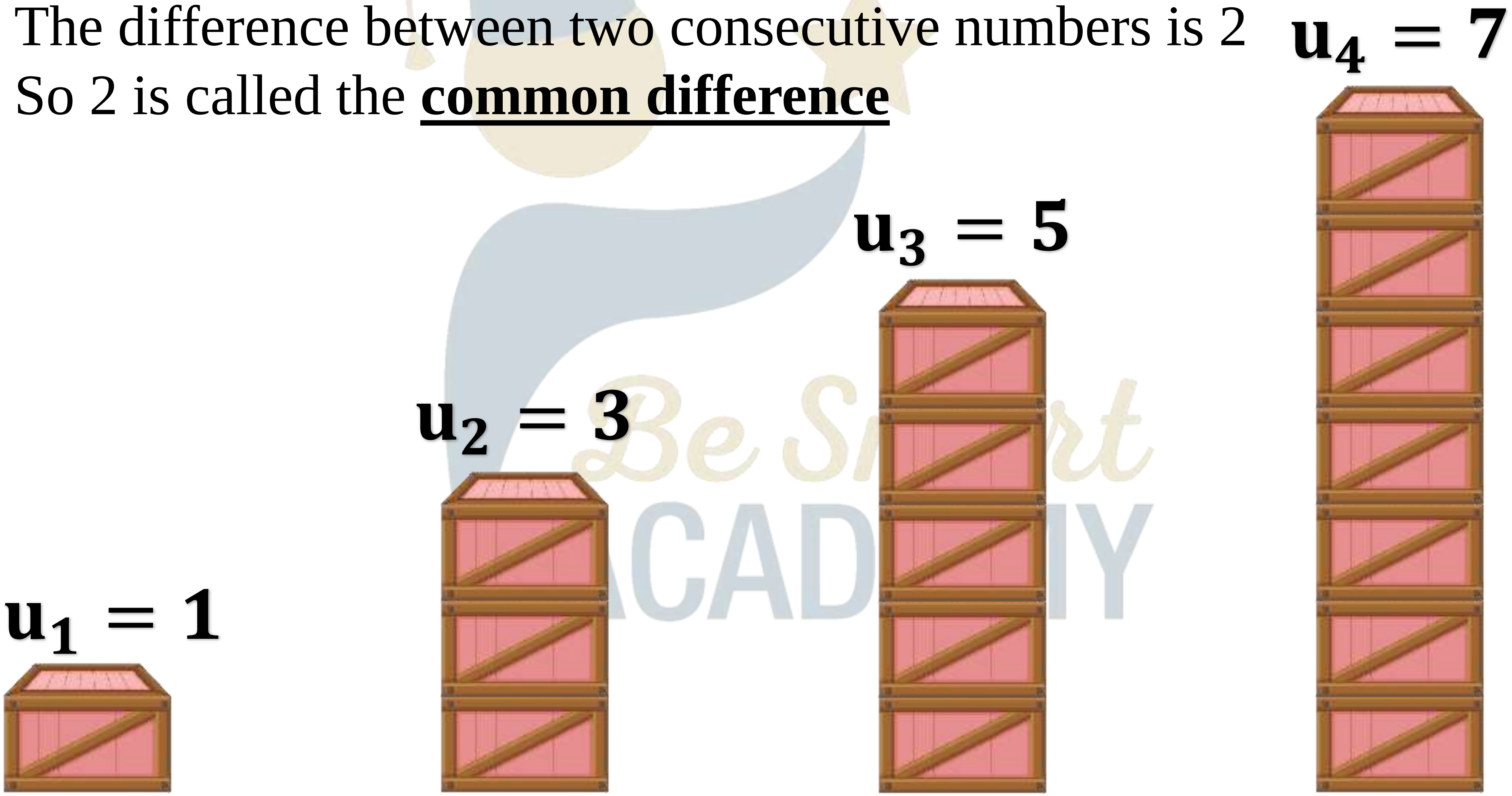
- n^{th} term: a_n or u_n …

3 Arithmetic sequence

Imagine that you have a box and you will add each time two boxes.

The difference between two consecutive numbers is 2
So 2 is called the **common difference**



3 Arithmetic sequence

$u_1 = 1$

$u_2 = u_1 + 2$

$u_3 = u_2 + 2 = u_1 + 2 + 2 = u_1 + 2 \times 2$

$u_4 = u_3 + 2 = u_1 + 2 \times 2 + 2 = u_1 + 3 \times 2$

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$u_n = u_1 + (n - 1) \times 2$

$\mathbf{u_1 = 1}$ $\mathbf{u_2 = 3}$ $\mathbf{u_3 = 5}$ $\mathbf{u_4 = 7}$

3 Arithmetic sequence

General term: $\mathbf{u_n = u_1 + (n - 1) \times d}$

where d is the common difference

The general term is used to calculate any term of the sequence:

$u_{10} = u_1 + (10 - 1) \times 2 = 19$

$u_{101} = u_1 + (101 - 1) \times 2 = 201$

3 Arithmetic sequence

Properties of Arithmetic sequence:

- **How to prove that a sequence is arithmetic?**

Show that the difference between any two consecutive general terms is constant: $u_n - u_{n-1}$ or $u_{n+1} - u_n$

- **What is the relation between any two terms u_p and u_q $p < q$?**

$$u_q = u_p + (q - p) \times d$$

- **What is the sum of the first nth terms?**

$$S_n = u_1 + u_2 + u_3 + \cdots + u_n$$

$$S_n = \frac{\text{number of terms}}{2}(\text{first term} + \text{nth term}) = \frac{n}{2}(u_1 + u_n)$$

3 Remark

If the sequence start by u_0 and the general term is u_n, the number of terms becomes n+1.

In general, the number of terms from one term to another is:
The difference between the positions of the two terms + 1
i.e. the number of terms from u_p to u_q is: $q - p + 1$
From u_0 to u_n is n-0+1=n+1 terms

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Application 1

You deposit $1 000 in a bank account that earns simple interest at a rate of 5% per year. What will your account balance be after 14 years?

Each year, the amount of money in the bank increased 5% of 1 000$, so this is an arithmetic sequence of common difference

$d = \frac{5}{100} \times 1000 = 50$ and first term $u_0 = 1\,000$

The general term is $u_n = u_0 + (n - 0) \times d = 1000 + 50n$

After 14 years: $n = 14$; $u_{14} = 1000 + 50 \times 14 = 1\,700$ $

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Geometric sequence

Imagine that you deposit in the bank $1 000.
Each year you earned 5% of your current balance.

Rank of the year	Balance in $
0	$u_0 = 1000$
1	$u_1 = 1.05 \times 1000 = 1050$
2	$u_2 = 1.05 \times 1050 = 1102.5$
3	$u_3 = 1.05 \times 1102.5 = 1157.625$
4	$u_4 = 1.05 \times 1157.625 = 1215.50625$

$u_1 = 1.05u_0$

$u_2 = (1.05)^2u_0$

$u_3 = (1.05)^3u_0$

$u_4 = (1.05)^4u_0$

4 Geometric sequence

General term: $u_n = u_0 \times r^n$

Where r is called the common ratio

The general term is used to calculate any term of the sequence:

$u_{10} = u_0 \times r^{10} = 1000 \times 1.05^{10} = 1628.8946$

$u_{101} = u_0 \times r^{101} = 1000 \times 1.05^{101} = 138076.3207$

4 Geometric sequence

Properties of Geometric sequence:

- **How to prove that a sequence is Geometric?**

Show that the ratio between any two consecutive general terms is constant: $\frac{u_n}{u_{n-1}} = r$ or $\frac{u_{n+1}}{u_n} = r$

- **What is the relation between any two terms u_p and u_q $p < q$?**

$$u_q = u_p \times r^{q-p}$$

- **What is the sum of the first nth terms?**

$$S_n = u_1 + u_2 + u_3 + \cdots + u_n$$

$$S_n = \text{first term}\left(\frac{1-r^{\text{number of terms}}}{1-r}\right) = u_1\left(\frac{1-r^n}{1-r}\right)$$

4 Application 2

Consider the sequence (u_n) defined

by: $\begin{cases} u_0 = 3 \\ u_{n+1} = \frac{u_n}{1-u_n} \end{cases}$ (assuming $u_n \neq 1$)

1. Show that (u_n) is not geometric nor arithmetic.

$u_0 = 3$

$u_1 = \frac{u_0}{1-u_0} = \frac{3}{1-3} = -\frac{3}{2}$

$u_2 = \frac{u_1}{1-u_1} = \frac{-\frac{3}{2}}{1+\frac{3}{2}} = \frac{-\frac{3}{2}}{\frac{5}{2}} = -\frac{3}{5}$

$\left.\begin{array}{l} u_1 - u_0 = -\frac{3}{2} - 3 = -\frac{9}{2} \\ u_2 - u_1 = -\frac{3}{5} + \frac{3}{2} = \frac{9}{10} \end{array}\right\}$ $u_1 - u_0 \neq u_2 - u_1$ so the sequence is not arithmetic.

$\left.\begin{array}{l} \frac{u_1}{u_0} = -\frac{\frac{3}{2}}{3} = -\frac{9}{2} \\ \frac{u_2}{u_1} = -\frac{\frac{3}{5}}{\frac{3}{2}} = \frac{-2}{5} \end{array}\right\}$ $\frac{u_1}{u_0} \neq \frac{u_2}{u_1}$ so the sequence is not geometric.

4 Application 2

Consider the sequence (u_n) defined

by: $\begin{cases} u_0 = 3 \\ u_{n+1} = \frac{u_n}{1-u_n} \quad \text{(assuming } u_n \neq 1) \end{cases}$

2. Consider the sequence (v_n) defined by $v_n = \frac{1}{u_n}$.
   a. Show that (v_n) is an arithmetic sequence.
   b. Determine its common difference and its first term.
   c. Express v_n in terms of n.
   d. Deduce u_n in terms of n.
   e. Calculate $S_n = v_0 + v_1 + \cdots + v_n$

a. $v_{n+1} = \frac{1}{u_{n+1}} = \frac{1-u_n}{u_n}$

$v_{n+1} - v_n = \frac{1-u_n}{u_n} - \frac{1}{u_n} = \frac{-u_n}{u_n} = -1$ constant

So (v_n) is an arithmetic sequence.

b. Common difference is d=-1

First term is $v_0 = \frac{1}{u_0} = \frac{1}{3}$

c. $v_n = v_0 + nd = \frac{1}{3} - n$

d. $u_n = \frac{1}{v_n} = \frac{1}{\frac{1}{3}-n} = \frac{3}{1-3n}$

e. $S_n = \frac{n+1}{2}(v_0 + v_n) = \frac{n+1}{2}\left(\frac{1}{3} + \frac{1}{3} - n\right) = \frac{n+1}{2}\left(\frac{2}{3} - n\right)$

4 Application 3

Consider the sequence (u_n) defined by: $\begin{cases} u_0 = 2 \\ u_{n+1} = \frac{1}{2}u_n + 3 \end{cases}$

1. Show that (u_n) is not geometric nor arithmetic.

$u_0 = 2$

$u_1 = \frac{u_0}{1-u_0} = \frac{1}{2}u_0 + 3 = 4 \quad ; \quad u_2 = \frac{1}{2}u_1 + 3 = 5$

$\left.\begin{array}{l} u_1 - u_0 = 4 - 2 = 2 \\ u_2 - u_1 = 5 - 4 = 1 \end{array}\right\} u_1 - u_0 \neq u_2 - u_1$ so the sequence is not arithmetic.

$\left.\begin{array}{l} \frac{u_1}{u_0} = \frac{4}{2} = 2 \\ \frac{u_2}{u_1} = -\frac{5}{4} \end{array}\right\} \frac{u_1}{u_0} \neq \frac{u_2}{u_1}$ so the sequence is not geometric.

4 Application 3

Consider the sequence (u_n) defined by: $\begin{cases} u_0 = 2 \\ u_{n+1} = \frac{1}{2}u_n + 3 \end{cases}$

2. Consider the sequence (v_n) defined by $v_n = u_n - 6$.
   a. Show that (v_n) is a geometric sequence.
   b. Determine its common ratio and its first term.
   c. Express v_n in terms of n.
   d. Deduce u_n in terms of n.
   e. Given $T_n = v_0 + v_1 + \cdots + v_n$. Calculate T_n in terms of n.

a. $v_{n+1} = u_{n+1} - 6 = \frac{1}{2}u_n + 3 - 6 = \frac{1}{2}u_n - 3$

$\frac{v_{n+1}}{v_n} = \frac{\frac{1}{2}u_n - 3}{u_n - 6} = \frac{\frac{1}{2}(u_n - 6)}{u_n - 6} = \frac{1}{2}$ constant

So the sequence is geometric.

b. Common ratio $r = \frac{1}{2}$

First term is $v_0 = \frac{1}{2}u_0 - 3 = -1$

c. $v_n = v_0 \times r^n = -1 \times \left(\frac{1}{2}\right)^n = -\frac{1}{2^n}$

d. $u_n = v_n + 6 = 6 - \frac{1}{2^n}$

e. $T_n = v_0\left(\frac{1 - r^{n+1}}{1 - r}\right) = -1 \times \left(\frac{1 - \frac{1}{2^{n+1}}}{1 - \frac{1}{2}}\right) = -2\left(1 - \frac{1}{2^{n+1}}\right)$

Summary

	ARITHMETIC	GEOMETRIC
General form	$u_n = u_1 + (n-1)d$ $u_n = u_0 + nd$	$u_n = u_1 \times r^{n-1}$ $u_n = u_0 \times r^n$
Proof	$u_{n+1} - u_n = d$ (constant)	$\frac{u_{n+1}}{u_n} = r$ (constant)
Sum $S_n = u_1 + u_2 + \cdots + u_n$	$S_n = \frac{n}{2}(u_1 + u_n)$	$S_n = u_1 \times \left(\frac{1-r^n}{1-r}\right)$

Remark: the number of terms from u_p to u_q is : $q - p + 1$

5 Variations of a sequence

General method:

Calculate $u_{n+1} - u_n$ (difference between two consecutive general terms)

- $u_{n+1} - u_n > 0$: (u_n) is increasing strictly
- $u_{n+1} - u_n < 0$: $(u_n)$is decreasing strictly
- $u_{n+1} - u_n = 0$: $(u_n)$is constant

Example: consider the sequence $u_{n+1} = \frac{n+1}{2n} u_n$; $u_1 = \frac{1}{2}$

Suppose that $u_n > 0$ for all $n \geq 1$

$u_{n+1} - u_n = \frac{n+1}{2n} u_n - u_n = \frac{-n+1}{2n} u_n$

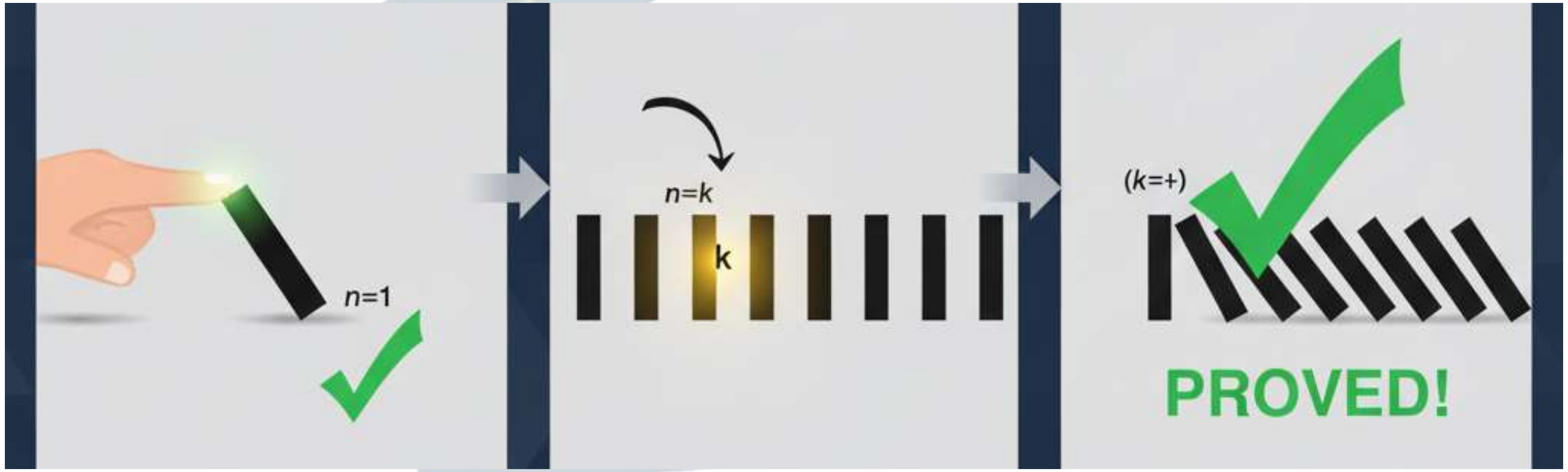
$n \geq 1 \Longrightarrow 1 - n \leq 0$ and $2n > 0$

$u_n > 0$ (given) so $u_{n+1} - u_n < 0$ then the sequence is decreasing strictly.

6 Mathematical induction

It is a method to prove statements that cannot be demonstrated by a direct argument. This proving technique can be compared to the process of making dominoes fail.

6 Mathematical induction



Step 1:
Show that the statement is true for the first value.

Step 2:
Assume that the statement is true for some arbitrary integer k.

Step 3:
Prove that it is true for k+1

6 Mathematical induction

Example: Consider the sequence $\begin{cases} u_0 = 1 \\ u_1 = 3 \\ u_{n+1} = 4u_n - 3u_{n-1} & \text{for } n \geq 1 \end{cases}$.

Show that for all n>0 that $u_n = 3^n$

For n=0: $u_0 = 1 = 3^0$

Suppose that the statement is true for n: $u_n = 3^n$

$u_{n+1} = 4u_n - 3u_{n-1}$

$= 4 \times 3^n - 3 \times 3^{n-1}$

$= 4 \times 3^n - 3^n$

$= 3^n(4 - 1)$

$= 3^n \times 3$

$= 3^{n+1}$ so $u_n = 3^n$ for all n